

Taking a Loop Walk Through the Integer Forest

A Coordinate-Free Holonomy Instrument for Multiplicative Structure

“Ah, but a man’s geometric and mathematical reach should exceed his grasp, or what’s a heaven for?” — after Robert Browning, Andrea del Sarto (adapted by the author)

Matthew “Bear” Schulz

Independent Researcher · Belmont, CA · ORCID: 0009-0000-0956-5743 mattschulz97@protonmail.com

Preprint — not peer reviewed. Priority established by timestamp. Released CC BY 4.0. v2, 2026-08-22 (assembled 2026-08-17, census sentence corrected in audit 2026-08-22). First deposited 2026-07-18. All figures regenerate from the companion receipt crate (sha-pinned).

WHAT CHANGED IN THIS VERSION

No measured value in this paper has changed. Every number in the first edition was computed by the code and every number was right. What was wrong was one sentence of prose describing the code, one worked example that followed from it, and one figure’s label placement. Those are corrected here, in the text, so that a reader who re-implements from this document reproduces the values it prints — which the first edition did not permit.

The construction text described the wrong climb. Section 2, step 3 said the walk climbs to a height equal to the running count of distinct primes used. The code climbs to the *rung of the largest prime factor encountered so far*, and only that reading reproduces the values printed in Sections 3 and 6: $\text{reach}(113) = 30.00110$ and $\text{reach}(100) = 4.26423$, against 1.03 and 3.63 under the running-count reading. The abstract carried the same error in compressed form. Both are corrected.

The worked example followed the wrong rule and is corrected with it. The $n = 100$ example in Section 2 said “climb one rung” once. Under the correct rule the walk climbs to $\text{rung}(2) = 1$ after the 2-leg and to $\text{rung}(5) = 3$ after the 5-leg, and it is the height of 3 that produces the printed 4.26423. This correction is not in the erratum of 2026-08-07; it was found while applying that erratum, because a fixed rule two lines above an unfixed example is still a document that cannot be reproduced.

Two different quantities were both called “the hypotenuse.” The flat chord $C(n) = \sqrt{x^2 + y^2}$, the distance home ignoring height, is what the closure deficit uses — and it is what makes $D(n) \geq 0$ true. The full reach $R(n) = \sqrt{x^2 + y^2 + \text{rung}^2}$, the distance home in three dimensions, is what the octave law uses. They are now defined separately in Section 2 and each site names which one it means.

Figure 4’s labels collided. Three labels overlapped in the lower left of the rendered figure. The corrected rendering — same content, same colours, labels repositioned — replaces it here.

A fourth correction was added in the 2026-08-22 build, found in independent audit. The ancestry census sentence in Section 5 credited all 13,197 on-core events to the walk-prefix predicate, while the receipt’s own table splits them: 13,187 walk grazings, all walk-prefix, plus 10 crossings by the loop’s return leg — a second mechanism the sentence did not name. Section 5 now states the split, with the ten events listed. No number changed; the sentence now shares a domain with its census.

The erratum record of 2026-08-07 remains posted as the dated provenance of these corrections. It is no longer the only place a reader can find them.

Abstract

We present a single coordinate-free construction — a compass walk driven by the prime factorization of an integer — that renders several classical objects of multiplicative number theory as exact, visually checkable geometry. Each integer n is mapped to a dead-reckoning path: from a north-pointing stem, one straight leg per prime factor p^e , of length e , at a heading equal to (the index of p among all primes) times the golden angle $137.50776\dots^\circ$, climbing to the rung of the largest prime factor encountered so far. The construction is injective on $[1, 10^6]$ with zero collisions. Within it, three classical quantities acquire exact geometric form: the total-prime-factor function $\Omega(n)$ appears as the walk's arc length ($\Omega = \text{walk} - 1$); the maximal chains of the divisor lattice appear as the axes the walk grazes (an “ancestry” relation with zero unexplained cases across 13,197 tested); and the torus knots $T(p,q)$ appear as the closed orbits of the construction's commensurate quotient, with $\det T(2,q) = q$ placing the trefoil, cinquefoil, and higher anchors on the odd integers. We do not claim new theorems; every core object is classical and is cited as such. Our contribution is methodological: one holonomy instrument, applied identically across substrates (here the integers; elsewhere geodetic, astrometric, and crystallographic data), in which the signal of interest is a quantized topological invariant rather than a metric magnitude. We give the construction, the measured regularities with their falsification receipts, one registered blind prediction, and an honest map of what is classical, what is an exact re-encoding, and what remains open pending specialist review.

0. Origin: A Loop Walk Through Many Forests

This section is the author's, in the author's voice. It is provenance, not proof: it explains where the construction came from and why the reader should trust the method rather than the speculation around it. Every claim that carries weight is in the sections that follow, each with a receipt.

My geometric reach has always exceeded my mathematical grasp. I cannot perform a derivative or an integral to save my life. What I can do is see mathematics as engineering components — parts to assemble into a larger structure and then validate by trying to break it. This paper is the result of that habit, applied for years across problems that looked unrelated until the same shape kept surfacing.

It began with whales. We were modeling how a diving hunter returns to a moving pod in the deep dark — quaternion tracks, dead-reckoning, a collapsing function that turns a foraging path into a straight-line return, and passive verification cheap enough to run without giving away position. The same questions reappeared at a poker table, where each player is a Byzantine sensor and the dealer watches a loop of wagers close around a hand. They reappeared again in crystal lattices as a memory substrate, in cooperative sensor arrays and gradiometers, and in the evolutionary problem of trusting a compromised map. Every time, the structure underneath was the same: a loop walked through some space, a holonomy read out at closure, and a substrate whose fidelity you check by watching where the walk fails to come home.

Some of the questions that pulled me were ones I could not close, and they belong here as questions, not answers. I kept circling repulsion and the electron shell — the way a *full* shell is a block, but the state one below full, $n-1$, is the one that can pass traffic and flex, the way a single vacancy lets a crystal lattice carry current and bend. That felt like the same optimized, near-full interaction state I kept meeting elsewhere, and it is, I think, the intuition that finally took form as the rungs, levels, and walls of the integer forest below. I record it as a pull, not a proof; whether an $n-1$ law lives in this construction is a question I have not yet tested.

The integers were the place all of it finally became exact. Instead of modeling a whale or a card table, I let each integer walk its own factorization: a heading for each prime, a distance for each exponent, a climb for each new prime, and a hypotenuse that closes the loop back to the origin as both an address and a residual. When I plotted these walks and scaled the vertical axis logarithmically, structure I had only felt in the other domains stood up in front of me — towers, cylinders, walls, seen cleanly from the top and the side. At the time, ten thousand integers was where the picture was clearest to me: a thousand was too few to see the forest, and my early renderer turned a million into a formless blur. That blur, I later learned, was the renderer and not the structure — on a logarithmic vertical axis with point density mapped honestly, the higher counts resolve *most* clearly of all. As the count climbs from a thousand to ten thousand to a hundred thousand to a million, the cofactor cylinders sharpen from scattered dots into clean, coherent walls (Figure 5): the coherence builds with resolution. The structure is exact; it only draws more sharply as the count grows.

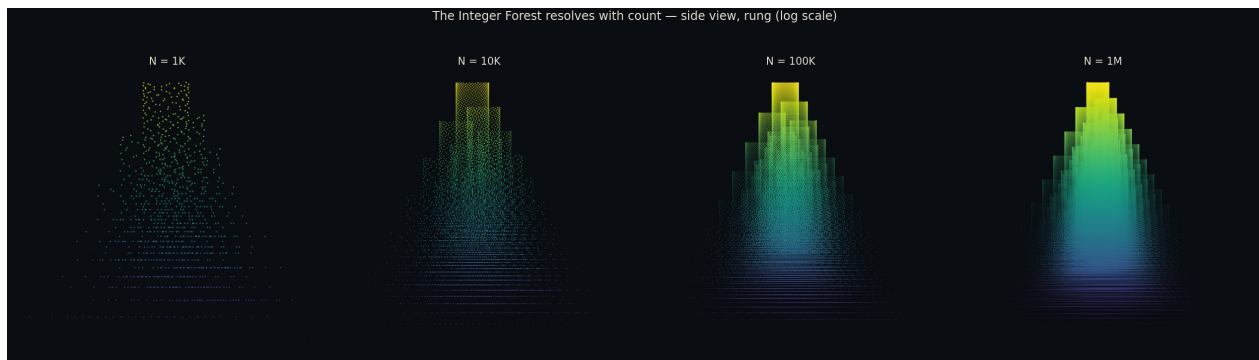


Figure 5. The forest resolves with count: side view (rung, log scale) at $N = 1,000$, $10,000$, $100,000$, and $1,000,000$. At a thousand integers the endpoints read as scattered dots; by a million the cofactor cylinders have resolved into clean, coherent walls, topping out at the 78,498 distinct-prime rungs below 10^6 . The structure is exact at every scale — higher counts draw it more sharply, not more noisily.

We rederived the torus and the trefoil from first principles before we learned they were long-established — and were glad to find we stood on firm, discovered ground (the crystallographic realization of factorization on a torus is due to Lulek and Olchawa; see Section 7). That is the spirit of this work: we are here because of everyone who published before us, and the theorems we kept rederiving from scratch were our validation landmarks, not our discoveries.

A note to the reader on what to trust. This construction has a lineage of exploratory notes and speculative cross-domain papers behind it. Those are exploratory by design and are not load-bearing here. If you want to test something, test *this* paper: the construction of Section 2, the measured regularities of Sections 3–7, and the registered prediction of Section 10, each of which regenerates from a sha-pinned receipt. Please judge the method by what is measured, not by the speculation it grew from.

1. Introduction and Motivation

Multiplicative number theory studies a handful of central objects: the primes as generators, unique factorization, arithmetic functions such as $\Omega(n)$ and $\omega(n)$, the divisor lattice, and smooth (friable) numbers. These are ordinarily studied algebraically or analytically. Visual approaches to the primes exist — the Ulam and Sacks spirals, and the golden-angle (Vogel) phyllotaxis spiral — but these index points by *position* n and read off patterns in the distribution of primes; they do not encode the *multiplicative* structure of each integer as a geometric path.

This paper takes a different route. Rather than plotting integers as points, we plot each integer as a *walk* whose steps are dictated by its factorization. The walk is a physical dead-reckoning: a heading for each prime, a distance for each exponent, a climb for each new prime. The result is that the arithmetic of n becomes the geometry of a path, and questions about factorization become questions about where paths go, what they enclose, and whether they close.

The purpose is not a new theorem about the integers. As we establish in Section 8 and document exhaustively in the companion prior-art analysis, every object we recover is classical. The purpose is an *instrument*: a coordinate-free way of turning multiplicative structure into holonomy — the failure of a loop to close, the winding of one path about another, the quantized invariants that survive when metric detail is discarded. The same instrument, applied to other substrates (satellite carrier-phase residuals, stellar kinematics, crystallographic orientation maps), reads the same kind of signal: an integer-valued topological quantity that is robust where raw magnitudes drift. The integers are the cleanest possible worked example, because here every invariant is exact rather than estimated through noise.

We are explicit about authorship from the outset. We invented none of the underlying objects — not the torus, not holonomy, not the golden angle, not the divisor lattice, not the torus knot. We used them, extensively; we rederived several from first principles to understand their pedagogy; and we then carried them, together, into a setting where they had not previously been assembled: a walkable geometric forest built from the prime factorization of the integers. The contribution is the assembly and the setting, not the parts. Prior art for each part is cited where it belongs, most directly the crystallographic realization of factorization on a torus due to Lulek and Olchawa [7], which we discuss in Section 7.

We state our epistemic standard plainly, because it shaped every result below. A quantity is reported as **measured** only when it is computed from an object that could have come out otherwise, and the target value is stated before the measurement. Regularities are **pre-registered** with their falsification conditions. Negative results are reported at equal weight with positive ones. Every figure in this paper regenerates from a sha-pinned receipt file in the companion crate.

2. The Construction

Let $n > 1$ have prime factorization $n = \prod p_i^{e_i}$. Define the **rung** of a prime p to be its index among all primes: $\text{rung}(2) = 1$, $\text{rung}(3) = 2$, $\text{rung}(5) = 3$, and so on. Let $\alpha = 360^\circ/\varphi^2 = 137.50776\dots^\circ$ be the golden angle, where $\varphi = (1+\sqrt{5})/2$.

The **path** of n is constructed as follows:

1. Begin at the observer, the origin $(0, 0, 0)$.
2. Take the **stem**: one unit step due north, to $(0, 1, 0)$.
3. For each prime factor $p_i^{e_i}$, in ascending prime order: turn to heading $\text{rung}(p_i) \cdot \alpha$ (measured in the horizontal plane), and walk e_i unit steps, while climbing to vertical height equal to the rung of the largest prime factor encountered so far.

The **address** of n is the endpoint of this path. Two different straight-line distances close that path, and the first edition of this paper used one letter for both; they are separated here and each later section names the one it means. The **flat chord** $C(n) = \sqrt{x^2 + y^2}$ is the distance home ignoring height — the closure deficit of Section 6 uses this one, and it is what makes that deficit non-negative. The **full reach** $R(n) = \sqrt{x^2 + y^2 + \text{rung}^2}$ is the distance home in three dimensions — the octave law of Section 3 uses this one. Measured anchors: $C(127) = 1.75547$ and $R(113) = 30.00110$. Because climbs move only the vertical coordinate and walks move only the horizontal coordinates, the construction cleanly separates a planar “compass” component from a vertical “smoothness” component.

Worked example ($n = 100 = 2^2 \cdot 5^2$): stem north one unit; two units at heading $\text{rung}(2) \cdot \alpha = 137.5^\circ$, with the height climbing to $\text{rung}(2) = 1$; then two units at heading $\text{rung}(5) \cdot \alpha = 3 \cdot 137.5^\circ = 52.5^\circ \pmod{360^\circ}$, with the height climbing to $\text{rung}(5) = 3$. Note that the climb is to the rung of the largest prime seen so far, not a step per prime: the final height is 3, not 2, and it is the 3 that produces the printed reach. The endpoint, the flat chord $C(100) = 3.03046$ and the full reach $R(100) = 4.26423$ are exact and reproduced to six decimals in the companion first-light receipt.

The construction of record is the compass walk defined above. An earlier quaternion-pose variant on the 3-sphere was tested and set aside for an unresolved intrinsic-axis ambiguity; it is retained in the archive marked superseded, and is not used here.

3. Injectivity and the Octave Law

Injectivity. The address map is injective on $[1, 10^6]$: a per-rung sorted closest-pair scan finds zero collisions across all four decades from 10^3 to 10^6 . Injectivity is falsifiable at every new decade; it holds to one million.

The octave law. For a prime p at rung r , the smallest power of two whose full reach exceeds $R(p)$ is $2^{\lceil r+1 \rceil}$. This “ $k = \text{rung} + 1$ ” law was pre-registered and holds at every tested rung out to rung 304, with a single exception at $p = 5$ (Section 6).

The cost of out-reaching a prime. The law prices a striking asymmetry between the two axes of the construction. To match the full reach of a single prime at rung r by a pure power of two requires $2^{\lceil r+1 \rceil}$, i.e. $2^{\lceil \pi(p)+1 \rceil} \approx 2^{\lceil p/\ln p \rceil}$ — the climb is *linear* in the rung while the smooth number needed to reach it is *exponential*. Concretely: the prime 113 sits at rung 30 with full reach $R = 30.001$, and the smallest power of two that exceeds it is $2^{31} = 2,147,483,648$ — a nineteen-million-fold difference in magnitude to buy one short step at that height. Equivalently, $729,000,000 = 2^6 \cdot 3^6 \cdot 5^6$ has a full reach of only 4.20, worse than 1000’s 5.367, because its legs nearly cancel. The reading: $R(n) \approx \max(\text{how rough } n \text{ is, how far it walked})$; roughness (measured in rungs) is cheap, walking (measured in exponents) is exponentially expensive. The primes form a reach that no smooth number of comparable size can attain — and this is precisely the roughness axis on which factoring algorithms operate.

These two facts establish that the construction is a faithful *coordinate* on the integers: distinct integers receive distinct addresses, and the vertical scale is organized by a clean doubling law. We emphasize — and return to this in Section 8 — that a faithful coordinate is the *opposite* of a cryptographic one: the address is trivially invertible (factor n , read the walk), so the construction offers no one-way hardness and no cryptographic content whatsoever.

4. The Cylinder Forest: Factorization as Nested Geometry

The central geometric fact of the construction is that **every integer lives on a cylinder of integer radius**.

For $n = m \cdot p^e$ with p the largest prime factor and m the cofactor, the horizontal address of n equals the horizontal address of m plus e unit steps at p ’s heading. Therefore n sits at horizontal distance **exactly** e — an integer — from a vertical axis planted at the endpoint of m . Across all 999,999 integers in $[2, 10^6]$, the worst deviation of this distance from the top exponent is 1.78×10^{-15} : floating-point roundoff. The radius is the exponent, exactly.

Figure 1 shows the whole construction at a glance — the foam and the towers in one $45^\circ/45^\circ$ frame.

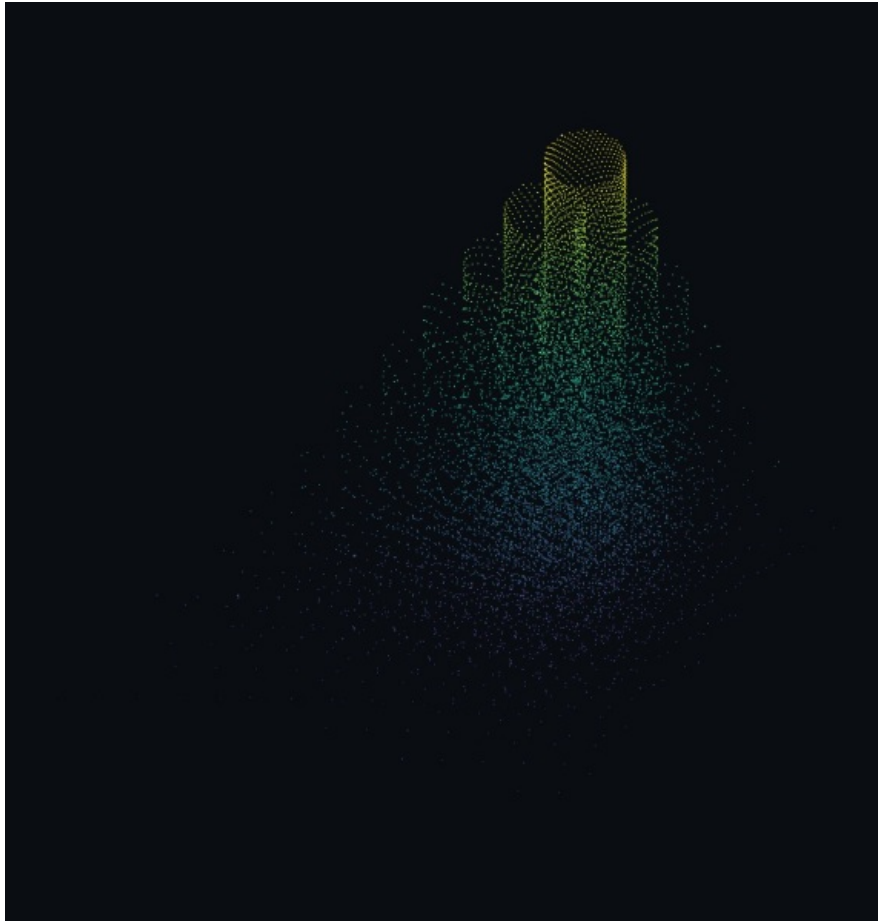


Figure 1. The integer forest at $N=10,000$, as a $45^\circ/45^\circ$ orthographic composite (azimuth 45° , elevation 45° ; rung on a log scale with $3\times$ vertical exaggeration, labeled). The dense low-rung “foam” and the taller cofactor cylinders appear together in one frame — the top-view circle structure and the side-view tower walls are two shadows of the same object. Regenerates from `tower_of_primes.py`.

Figures 2 and 3 show it at $N=10,000$ from two orthogonal angles: the top view (down the vertical axis) reveals the cylinders as a foam of circles — each a locus of endpoints for a fixed cofactor as its top prime sweeps the golden angle — while the side view reveals the same cylinders as parallel walls, sharpened by arcsine limb-brightening at the silhouette edges.

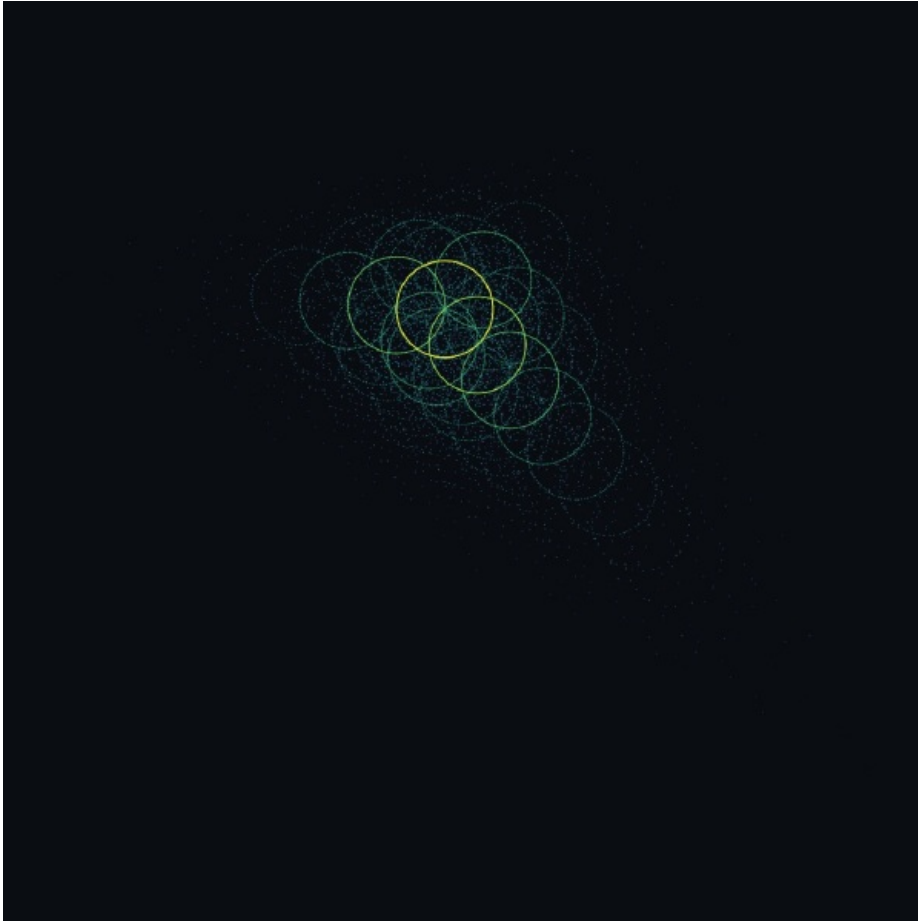


Figure 2. Top view (x, y) of the tower at $N=10,000$, looking down the vertical axis. Integers appear as a foam of unit circles; the gold ring marks the prime cylinder C_1 at radius 1, which carries all 1,229 primes below 10,000. Regenerates from the receipt code.

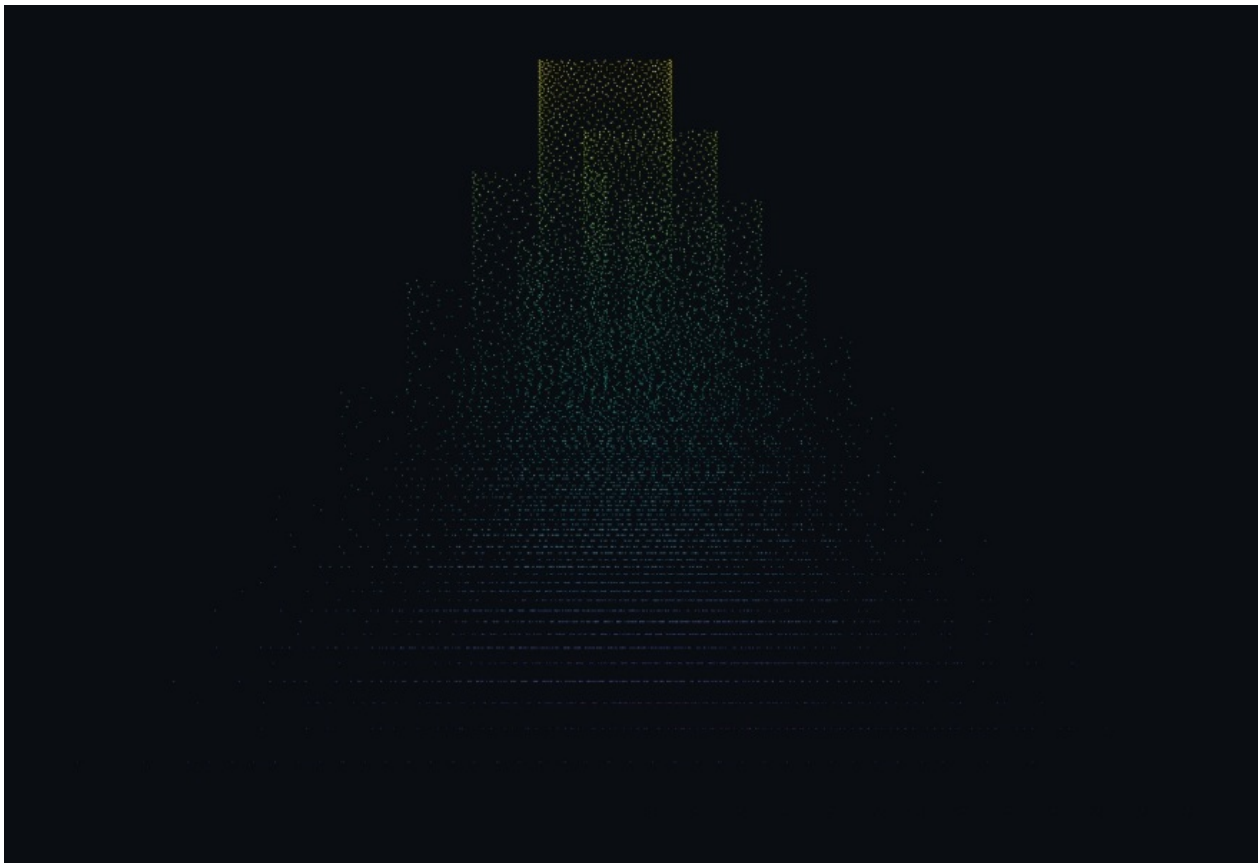


Figure 3. Side view (x, rung, log scale) of the same 10,000 integers. The cylinders appear as parallel vertical walls; the stepped skyline is the smooth-number filtration by rung.

Figure 4 is a schematic of the nesting anatomy.

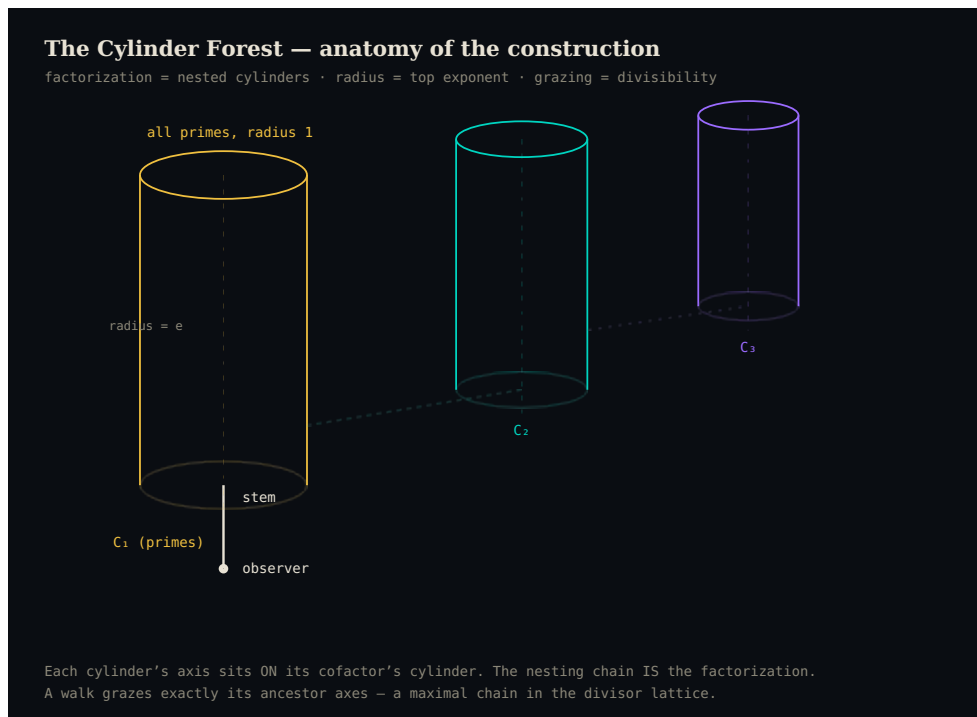


Figure 4. Schematic anatomy of the cylinder forest: the stem, the prime cylinder C_1 at the stem tip, and nested cofactor cylinders whose axes each sit on the cylinder of their own cofactor. The nesting chain is the factorization. — label positions corrected in v2; the drawing, the colours and every label's text are unchanged from the first edition.

A census of these radii shows that **99.09%** of integers in $[2, 10^6]$ live on a *unit* cylinder (top exponent 1). The dominant cylinder is the prime cylinder C_1 , axis at the stem tip (0, 1), carrying every prime — since each prime’s path is the stem plus a single unit step, all primes lie at radius exactly 1 from that axis.

The cylinders nest recursively: the axis of C_m is itself a point on the cylinder of m ’s own cofactor, and so on down to the stem. **The nesting chain is the factorization.** Read geometrically, an integer’s address is: which cylinder, on which cylinder, on which cylinder, rooted at 1. This is a metric realization of the divisor tower of n .

The vertical structure of each cylinder wall carries a measurable decomposition. Treating the closed loop (walk out, straight line home) as a circuit and computing its vector area $S = \frac{1}{2} \oint \mathbf{r} \times d\mathbf{r}$ — the Stokes object whose flux through any named field would be a holonomy — the area tracks the full reach closely (for a prime it is a tall thin sliver of base 1 and height equal to the climb, so $|S| \approx$ the rung). Crucially, along the walls the connection is *flat*: the golden-angle offsets all share the vertical axis and therefore commute, so the transport is abelian and carries no holonomy of its own. This is why the construction’s only closure defect lives in the loop as a whole (Section 6) and not on the walls — the walls are curvature-free by measurement, and the vertical (rung) component of the loop’s vector area is the decomposed z-residual of the walk. Whether transport across cylinder *boundaries* (caps and bases, where a walk enters or leaves a cylinder) carries holonomy is not measured here and is left open.

5. The Ancestry Theorem: Grazings as Divisor Chains

The cylinder forest raises a precise question: which axes does a given integer’s walk *touch*?

Statement. The walk of n grazes (passes at horizontal distance zero from) the axis of m if and only if m is a **walk-prefix** of n — an initial segment of n ’s factor-walk, including partial exponents: $m = (\prod_{i < j} p_{-i}^{e_{-i}}) \cdot p_j^c$ for some $1 \leq c \leq e_j$.

Measurement. Across all integers $n \leq 10^4$ and cofactor axes $m \in \{2, \dots, 10\}$, the census counts 13,197 on-core events, and they split into two named mechanisms. **13,187 are grazings by the walk itself, and every one is a walk-prefix, with zero unexplained** — the statement above, exact over its domain. The remaining **10 are crossings by the loop’s return leg**, the straight line home, which passes through an axis point the walk itself never touches: once at the axis of 6 (by $n = 3$), nine times at the axis of 7 (by $n = 21, 63, 189, 567, 630, 1701, 1890, 5103, 5670$). Every event is explained; the iff above governs the walk, and the return-leg crossings are the closed loop’s own, separately verified, contribution.

Interpretation. The axes a walk grazes form a chain $1 \mid d_1 \mid d_2 \mid \dots \mid n$ in the divisor lattice — a **maximal chain**, of length $\Omega(n)$, in the chosen prime order. The number of such chains is the multinomial $\Omega(n)! / \prod e_i!$. Thus the ancestry structure is a geometric realization of the maximal chains of the divisor lattice: divisibility becomes tangency, and a loop’s “ancestors” become exactly the axes it touches.

This is, to our knowledge (Section 8), the cleanest statement in the construction and one of its two candidates for a genuinely new *encoding* of a classical object.

6. Measured Regularities

We record the principal pre-registered measurements. Each has a falsification receipt in the crate.

Walk length is $1 + \Omega(n)$. The total arc length of the walk equals 1 (stem) plus $\Omega(n)$, the total number of prime factors counted with multiplicity — verified exactly across the tested range. The walk therefore carries the classical function Ω ; the geometric content beyond Ω lives entirely in the *flat chord* $C(n)$, a two-dimensional golden-angle character sum over the prime factors (Section 7).

Two routes to one address; the closure deficit. Every integer’s endpoint can be reached two ways: the *long walk* (the scenic route through the towers, of length $1 + \Omega(n)$) and the *flat chord* (the straight-line collapsing function $C(n)$, the horizontal distance from endpoint to origin). Same address, two costs. Their difference $D(n) = (1 + \Omega(n)) - C(n) \geq 0$ is the **closure deficit** — and it is the chord rather than the full reach that appears here, because the walk and the chord lie in the same plane, so the triangle inequality forces the sign. Against the full reach the difference goes negative for almost every integer, which is the octave law read backwards rather than a deficit — the toll paid for not closing the loop directly. It is near zero for a prime (one straight leg: nothing to buy back) and large for a highly composite number whose legs zig-zag. This is one of the two candidate re-encodings discussed in Section 8: the walk length is the classical Ω , but the chord is a golden-angle character sum whose specific form we have not located in the literature.

The comma: structural kin, not a numerical bridge. The closure defect has a familiar ancestor in the substrate from which this program’s holonomy language first came. The Pythagorean comma — the 23.46-cent gap ($3^{12}/2^{19}$) by which twelve just fifths overshoot seven octaves — is the musical instance of the same phenomenon the construction measures: a loop of commensurate-but-unequal ratios that fails to close, i.e. a holonomy. It is the reason the circle of fifths is a spiral rather than a circle. We record the kinship at the level of *kind*, not number. The comma is the same type of object as the closure deficit — a loop that does not return to its origin — and it is the commensurate-substrate cousin of the incommensurable non-closure that makes the golden helix never repeat (Section 7). But its value does not bridge to any

tower quantity: we tested the numerical correspondence between the comma and the construction's measured constants and report it negative. The comma enters here as a second worked instance of closure failure and as provenance for the method, not as an identity.

Stiff spreading. The mean horizontal distance of an integer's endpoint from the stem grows with $\Omega(n)$ as a power law with exponent $\alpha = 0.977$ — nearly ballistic, decisively not the 0.5 of a diffusive random walk. The construction is stiffer than a random walk because repeated prime factors contribute perfectly correlated (collinear) steps. A pre-registered diffusive ($\sqrt{}$) reading was thereby falsified and is reported as such.

Frame-lock. The vector sum of all rotor headings over the first 78,498 rungs never exceeds magnitude **1.072924**, which equals the theoretical ceiling $1/\sin(\alpha/2)$ to six decimals. The sum passes through near-zero “tune points” at rung counts 233, 377, 610, 987 — all Fibonacci — and also at 754 and 843, which are sums of two Fibonacci tune points ($754 = 2 \cdot 377$, $843 = 233 + 610$). Operating at Fibonacci-length windows minimizes net rotational drift, consistent with the low-discrepancy behavior of golden-angle sequences.

Fibonacci returns. The near-returns of the heading (the values of $\Delta\text{runc} \cdot \alpha \bmod 360^\circ$) occur at $\Delta\text{runc} = 1, 2, 3, 5, 8, 13, 21, 34$ — Fibonacci, with alternating sign, i.e. the continued-fraction convergents of φ^2 . The pentagon ($\Delta\text{runc} = 5$) is the $5/2$ convergent.

Liouville bias. In $[2, 10^6]$, integers with an odd number of prime factors exceed those with an even number by 531 — a small persistent parity bias. We note for honesty that the analogous Pólya bias is known to reverse at $n = 906,150,257$; our window lies entirely before that reversal, and no claim of a persistent bias is made.

The $p = 5$ anomaly. The octave law $k = \text{runc} + 1$ has exactly one exception among primes tested to 17,389: $p = 5$. The cause is measured and named: 5 sits at rung 3, whose heading (52.5°) is the most strongly northward of any low rung ($\cos = +0.608$); this vertical boost lifts $R(5)$ just past the 2^4 rung, requiring 2^5 . No larger prime, and neither lower rung, has both the sign and magnitude to jump an octave step. $p = 5$ is the unique golden-heading/octave resonance. It is worth noting that this same integer reappears in the commensurate quotient (Section 7) as the determinant of the cinquefoil torus knot, $\det T(2,5) = 5$: the one prime that breaks the octave law and the determinant of the pentagon knot are the same number, the anomaly and the invariant arising from the same golden-angle structure that gives rung 3 its northward heading.

7. The Commensurate Quotient: Torus Knots and the Rotation-Number Dial

The golden angle is irrational as a fraction of a turn: $\alpha/360^\circ = 1 - 1/\varphi^2 = 0.381966\dots$. Consequently the heading never repeats, the helix on each cylinder is dense and never closes, and there are no standing modes — the construction has no torus as a closed orbit.

Replacing the golden angle by a *commensurate* angle $2\pi/n$ changes this completely. The rotation number becomes rational, orbits close, and the closed orbits are exactly the **torus knots and links $T(k, n)$** obtained by skip- k weaving on an n -gon. The number of components is $\gcd(k, n)$ (the weave law), the genus of a torus knot is $(p-1)(q-1)/2$, and the determinant satisfies $\det T(2, q) = q$ for odd q . This places the classical anchors on the odd integers: the trefoil $T(2,3)$ at determinant 3, the **cinquefoil $T(2,5)$ at determinant 5**, the septafoil $T(2,7)$ at determinant 7. Every such torus knot is chiral; the parity/handedness bit lives in the writhe sign.

Thus the golden construction and its commensurate quotient are the two ends of a single **rotation-number dial**: irrational (dense, non-closing, no resonances) versus rational (closing, resonant, knotted). This dial is precisely the distinction studied classically in torus-knotted Reeb dynamics, where the rotation number $p/q + \delta$ interpolates between commensurate closure and an irrational perturbation. Our contribution here is only the identification: the arithmetic construction sits at the maximally irrational (golden) end, which is *why* it never closes and *why* its commensurate quotient produces the knot anchors.

We note prior art on the general idea of realizing factorization as motion on a torus. Lulek and Olchawa [7] realize a finite cyclic chain of length N as an n -dimensional toroidal crystal indexed by the n distinct prime divisors of N , with the Sylow decomposition serving as lattice translations along n mutually perpendicular directions (worked example $N = 36 = 2^2 \cdot 3^2$). Their construction and ours share the core idea — factorization becomes motion on a torus indexed by primes — and differ in the embedding: their axes are mutually perpendicular, one per distinct prime, while ours are golden-angle headings indexed by the rank of each prime among all primes. We claim no novelty for the factorization-to-torus idea itself; it is prior art.

8. What Is Classical, What Is an Exact Re-encoding, What Is Open

We are explicit about novelty, because the value of this work depends on not overstating it.

Classical (cited, not claimed). The golden-angle/Vogel spiral and prime phyllotaxis; the identity of $(\mathbb{Z}_{>0}, \times)$ as the free abelian group on the primes; the smooth-number filtration underlying the number field sieve; Wilson-loop/holonomy as a cross-domain probe unifying winding number, Chern number, and Hall conductance; torus knots $T(p,q)$ and their

invariants; maximal chains in the divisor lattice and their multinomial count; exponential sums twisted by multiplicative functions including Ω . Every regularity in this paper reduces to one of these.

Prior art on factorization-to-torus. The general idea that the prime factorization of an integer can be realized as motion on a torus is not new: Lulek and Olchawa [7] gave a crystallographic realization in 1996 (Section 7). We do not claim it.

Exact re-encodings (the two remaining candidates). After a credentialed database search (zbMATH Open), two objects were not located in this specific geometric form: (1) the **flat chord** $C(n)$ as a two-dimensional character sum whose frequencies are the golden angle indexed by *prime rank* — distinct both from the linear-phase and Dirichlet-character exponential sums of the classical theory, and from Lulek’s mutually-perpendicular per-prime axes; and (2) the **ancestry theorem**, the metric realization of divisor-lattice maximal chains as golden-angle walks in which axis-grazing is equivalent to divisibility. Both reduce to classical cores (Ω ; divisor chains); the candidate novelty is the *golden-angle-per-prime-rank encoding*, not the underlying object. These candidates are narrow and we assert them only as “not located,” not as proven new.

Open (pending specialist review). Whether either remaining re-encoding is genuinely new we leave open. Our search covered arXiv, published PDFs, and a credentialed pass through zbMATH Open [references below]; it did not include MathSciNet (subscription-gated). A definitive determination requires a referee in analytic number theory and one in lattice/order theory. We claim priority for the *construction and its measured regularities*, dated by this preprint; we claim no unproven novelty.

9. The Instrument Beyond the Integers

The reason to build this construction is that it is one instance of a portable methodology. The same instrument — loops, a connection, a quantized winding invariant, and a taxonomy of verdict states (regular / on-core / non-orientable) — applies wherever data can be walked in loops and a holonomy read out. In the integers the connection is the golden-angle rotation, the loops are integer walks, the invariant is the winding of a walk about a forest axis, and the on-core state has an exact arithmetic meaning (ancestry). In geodetic carrier-phase data, stellar kinematics, and crystallographic orientation maps, the same three verdict states carry domain-specific meaning, but the reading discipline is identical: trust the quantized channel, treat the singular set as a named state rather than an error, and pre-register before measuring.

The breadth is the thesis. No single result in this paper is load-bearing; the contribution is that one exact, checkable instrument recurs across substrates that are ordinarily studied in isolation. The integers are where the instrument can be checked to machine precision, which is why they are presented first. The instrument’s geodetic, stellar, and quasar first-lights are recorded in the companion deposit *Reading the Sky Through Loops* (DOI 10.5281/zenodo.21331652).

A structural observation supports this framing. The classical objects underlying this construction are distributed across mathematics-subject classifications that do not cross-reference: the golden-angle spiral as a lattice sits in Diophantine approximation (11J) and number-theoretic lattices [1]; the rotation-number dial in circle dynamics (37E) [6]; the factorization-torus in crystallographic groups (20H) [7]; phyllotaxis packing in discrete geometry (52C) and mathematical biology (92C) [8]. These are, in substantial part, the same underlying mathematics carrying different reviewer classifications, with no single classification assigned to see across them. The value of a coordinate-free instrument is precisely that it renders this shared structure in one picture. The gap it addresses is not one of coverage — each piece is deeply studied — but one of cross-reference.

10. A Registered Blind Prediction

To make the construction falsifiable in the strongest sense, we register a prediction before the relevant computation is run. The closest same-rung separation record — the smallest horizontal distance between two integers on the same rung — is 2.906927×10^{-4} at $N = 10^6$ (the pair $628254 = 2 \cdot 3^2 \cdot 11 \cdot 19 \cdot 167$ and 808781). Our kernel-vector model predicts that as N increases this record holds flat until exactly $N = 4,225,798$, at which point it drops to 8.346644×10^{-5} , realized by the pair $154,449 = 3^2 \cdot 131^2$ and $4,225,798 = 2 \cdot 127^2 \cdot 131$, and then holds through $N = 10^7$. The prediction, its search class, and its three pre-registered failure modes are sha-pinned in the companion crate. A subsequent computation to 10^7 , run without consulting the registered file, is the test.

11. Conclusion

We have shown that a single dead-reckoning construction turns the multiplicative structure of the integers into exact geometry: factorization becomes a nested cylinder forest, $\Omega(n)$ becomes arc length, the divisor lattice’s maximal chains become tangency events, and the commensurate quotient produces the classical torus-knot anchors on the odd integers. None of this is new mathematics; all of it is a new *lens*, and the lens is portable. We have been explicit about what is classical, what is a candidate re-encoding pending specialist review, and what is measured versus conjectured. We claim priority for the construction and its regularities, we claim no unproven novelty, and we release everything freely.

Interactive viewer: [code/tree_of_primes_1M.html](#) renders all of these views — top, side, front, and free 3D orbit — at 1K to 1M with a log/linear rung toggle, for readers who want to turn the construction in their own hands.

An Invitation

We ship this construction with its computations reproducible to the hash and its central prediction falsifiable by design. We do not claim it is beyond criticism; we claim the opposite — that it is built to be criticized efficiently. Because every figure regenerates deterministically from a sha-pinned receipt, disagreement need not be spent on whether the numbers are right (anyone can verify that in minutes) and can instead be aimed where it belongs: at the construction's choices and their meaning. Is the golden angle the right connection, or an arbitrary one? Does grazing-equals-divisibility reveal structure in the integers, or only in the embedding? Is the closure deficit measuring number theory, or measuring our coordinate system? These are the real questions, and the mathematics does not settle them — it only guarantees that the fight can happen on solid ground.

We therefore invite the field's hardest critics to attack the ideas, not the arithmetic, and we commit to logging every result — confirming or refuting — in the open, with receipts, so that the examination itself accumulates rather than evaporates. Each landing that cracks the object makes it more precise; each that rings true is recorded so the swing need not be repeated. That is the intended life of this work: not a claim to be defended, but a tool to be tested, in the open, by anyone, until what survives is known to have survived.

Epilogue: On Standing Beside

I stand beside, rather than on the shoulders of, the mathematicians, the tinkerers, and the engineers who came before me — who published their ideas and gave names to theorems and constructions. We always hoped we had broken new ground as we drove our tool across domains, but our only genuine novelty was the assembly, and a few ideas for visually displaying and working with the geometry. Everything load-bearing beneath it was already discovered, already named.

Each rederived theorem was a waypoint — a solid, established, zero-holonomy landing point, confirming we were not drifting into speculation without the structure to follow it. We landed blind from first principles, again and again, and each time found the ground already mapped and named by someone before us. We would stand a moment on that safe, explored ground to catch a breath and read the shape of the road signs pointing to the next direction — and the direction was always the same.

Most researchers hope for the novelty of pure discovery. We came to welcome the opposite: the sight of prior explorers in our same vein. Every established landmark we stumbled onto told us the tool was pointed at something real, because others, coming from entirely different domains, had stood in that same spot before us. The connections we saw between fields were connections that could have been seen before — and sometimes were, in one silo or another, by someone who did not know their neighbor two classifications over had seen the same thing.

We thank the scholars who came before us. This work is theirs as much as ours; we only walked the loop that connects their landmarks, and drew the map.

References

- [1] R. V. Jean, *Number-theoretic properties of two-dimensional lattices*, J. Number Theory **29** (1988), no. 2, 206–223. Zbl 0656.10007.
- [2] T. van Ravenstein, *Optimal spacing of points on a circle*, Fibonacci Quart. **27** (1989), no. 1, 18–24. Zbl 0665.10009.
- [3] H. Hellwig, T. Neukirchner, *Phyllotaxis: die mathematische Beschreibung und Modellierung von Blattstellungsmustern*, Math. Semesterber. **57** (2010), no. 1, 17–56. Zbl 1234.11170.
- [4] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge Studies in Advanced Mathematics **46**, Cambridge Univ. Press, 1995. Zbl 0831.11001. (Background: Ω , ω , exponential sums twisted by multiplicative functions.)
- [5] S. E. Graiff Zurita, R. Oishi-Tomiyasu, *Packing theory derived from phyllotaxis and products of linear forms*, Constr. Approx. **60** (2024), no. 3, 515–545. Zbl 1553.52010.
- [6] F. C. Motta, P. D. Shipman, B. D. Springer, *Optimally topologically transitive orbits in discrete dynamical systems*, Amer. Math. Monthly **123** (2016), no. 2, 115–135. Zbl 1351.37170. (Three-distance theorem; extremal approach-to-density of the golden rotation.)
- [7] T. Lulek, R. Olchawa, *Geometric interpretation of fractal symmetries of a finite linear chain*, J. Math. Phys. **37** (1996), no. 5, 2472–2483. Zbl 0868.20041. (Factorization realized as an n -dimensional toroidal crystal indexed by distinct prime divisors.)

[8] Additional classical background located by prior-art search — torus knots $T(p,q)$ and their invariants; maximal chains in the divisor lattice; the free abelian group on the primes; smooth-number filtration; Wilson-loop/holonomy as cross-domain probe; torus-knotted Reeb dynamics with rotation number $pq+\delta$ — is documented with sources in the companion crate files PRIOR-ART-GAUNTLET.md, NOVELTY-VERDICT.md, NOVELTY-DEEPSEARCH.md, and ZBMATH-PASS.md.

Reproducibility

Every numerical claim regenerates from a sha-pinned receipt file in the companion crate (crate_2026-07-17_VERIFIED.zip). Key receipts: baseline verification (VERIFY-RECEIPT), injectivity and octave law (EXT-RECEIPT), cylinder census (CYLINDER-RECEIPT), ancestry theorem (ONCORE-ANCESTRY), stiff spreading and Ω identity (ONCORE-ANCESTRY, TWO-OBJECTS-DEV), frame-lock and Fibonacci returns (FRAMELOCK-RECEIPT), anomaly hunt and $p=5$ (ANOMALY-HUNT), torus-knot loom spectrum (LOOM-SPECTRUM), the registered prediction (PREDICTION-10M), and the prior-art and novelty analyses (PRIOR-ART-GAUNTLET, NOVELTY-VERDICT, NOVELTY-DEEPSEARCH). All code is pure-standard-library and deterministic.

Acknowledgments

Developed in sustained collaboration with Claude (Anthropic) as computational bench and prior-art foil. All work released freely to humanity — no patents, no paywalls. CC BY 4.0.