

The Crossover Atlas

One object, many walls — a translation companion to the Field Guide and the Tower papers

Matthew “Bear” Schulz

· Independent Researcher · Belmont, CA · CC BY 4.0 · v3, 2026-08-22 (chapters 1–5 written 2026-08-09)

Preprint, not peer reviewed. v3 adds Chapters 6–8 (edge states; Gauss-Bonnet/Chern-Weil; Girard/Berry), reconciles every receipt pointer to one convention, brings the T05 and T11 spokes into the crate, and corrects one measured overstatement in Chapter 2 (the (84,53) row is $6.5\times$ tighter, not “an order of magnitude”).

Standing beside

This is an **expository** paper. It approaches known walls from new directions and, every time, names whose wall it is. The value here is the map drawn *between* the landmarks, not the landmarks — every result used is classical, or belongs to the field that built it, and is cited in the References. Novelty language is restricted throughout to “not located,” never “new.” The one genuinely internal contribution is small and stated where it occurs (Chapter 2). A credentialed zbMATH / MathSciNet pass has not yet been run; any located prior art will be logged in the open with attribution. We are here because of everyone who published the walls before us.

How to read this atlas

Each chapter walks one wall in a language it was not built in, and every chapter ends with a **verdict**, because a fluent translation hides whether anything is under it. The verdicts:

- **[exact / theorem-backed]** — the same object, with a receipt or a proof.
- **[settled / classical]** — established result, cited to its source.
- **[kinship of kind]** — the same *type* of object; any numerical bridge is tested and its result reported.
- **[conjectural]** — a resemblance that is measured but whose mechanism is unproven.
- **[inherited coincidence, flagged]** — a rhyme kept only so a reader is told it is a rhyme.

Every number in the chapters appears in a receipt (the .py files) that regenerates it from constants. No chapter asserts more than its tag allows.

What Losing a Dimension Buys You

Novikov Discounts at the Floer Market

Crossover Atlas, Chapter 1. A translation, not a discovery. Matthew “Bear” Schulz · CC BY 4.0 · 2026-08-09. Receipt: 02_receipts/spectral_flow.py, 02_receipts/flatland_walk.py.

How to read this atlas. Each chapter walks one wall in a language it was not built in, and every chapter ends with a verdict, because a fluent translation hides whether anything is under it. The verdicts are the ones this program has used all along: **[kinship of kind]** (the same type of object, numerical bridge tested and reported), **[exact re-encoding / theorem-backed]** (the same object, with a receipt or a proof), and **[inherited coincidence]** (a rhyme that collapses the moment a specialist looks, kept only because someone will notice it and should be told it is a rhyme). No chapter asserts more than its tag allows.

The wall

Floer homology is one of the deep instruments of modern geometry, and it is expensive. To define its differential at all you need Fredholm theory, transversality, Gromov compactness, gluing, and — in the hard cases — a virtual fundamental cycle to make the count well posed. That machinery is what lets it see exotic smooth structures, the Casson invariant, Donaldson’s four-manifold invariants: merchandise no other tool stocks. This chapter does not buy any of it. It walks into the same market, shops only the discount aisle, and reports exactly what is on the shelf there and exactly what is not.

The discount aisle is one dimension down, and it has a name

Floer homology is Morse homology in infinite dimensions [settled]. Drop the infinite-dimensionality — read a *circle-valued* Morse function instead of an action functional on a loop space — and you land in **Novikov homology**: Morse theory for a function valued in a circle, generators are critical points, the differential counts flow lines over the Novikov ring of formal series in the winding variable. Novikov is not a cheap imitation of Floer. It is the finite-dimensional prototype Floer generalises, *and* it survives inside Floer as its coefficient ring, because the symplectic action is circle-valued on loop space [settled]. So the discount aisle sells Floer’s own scaffolding — the grading, the chain complex, the coefficient bookkeeping — at abelian prices.

The modulus-1 sweep of the integer forest is a circle-valued Morse function. That is the honest address of the tower: **Novikov city, and Floer city is built on its foundations one floor up** [exact re-encoding].

The odd member of the family: wind the loop, read the flow

The flat $U(1)$ connections we walked in the companion rung sit still — you read their holonomy and stop. The odd member keeps $U(1)$ but lets the loop of connections *wind*, and reads what the spectrum does as it goes.

Take the circle S^1 threaded by a flat $U(1)$ flux a — holonomy $e^{(2\pi i \cdot a)}$. Couple it to the momentum operator; the spectrum is the integer lattice shifted by the flux, $\lambda_n(a) = n + a$. This is the Aharonov-Bohm ring exactly [settled]: thread flux, and the whole spectrum slides. Wind the flux once and the spectrum returns to itself shifted by one slot — one eigenvalue has crossed zero. Wind it w times and **the net eigenvalue flow across a fixed level is exactly w** :

```
winding w :    1    2    3    5   -1   -2    8
spectral flow: 1    2    3    5   -1   -2    8      (MEASURED, spectral_flow.py)
```

Spectral flow = winding number = the first Chern number over the mapping torus [exact / theorem-backed, Atiyah-Patodi-Singer]. No analysis, no gauge choice; $U(1)$ is abelian, so the whole thing is a counting problem.

(An honest note on the receipt, because the method demands it: the first two versions of the counter disagreed with the claim — an eigenvalue landing exactly on a sample point read as sign zero and was skipped. The table said NO while the prose wanted to say yes. That is this program’s own “prose ahead of the number,” and it was caught by committing the check before trusting the sentence. The shipped version uses a half-open crossing convention and a non-integer reference level, and it confirms.)

Why this is the thing worth carrying

Spectral flow is not a curiosity next to Floer homology — it *is* how Floer grades. The Floer index of a flow line is the spectral flow of the linearised operator along it [settled]. In the abelian corner that spectral flow collapses to a winding number you read off directly. So the tower’s “integer tick” — the winding of its modulus-1 climb — is the raw material of the Floer grading, computed for free because the group is abelian. That is the discount: **the $\mathbb{Z}$ -grading that Floer city runs on, bought as a counting problem instead of an operator-spectrum problem**.

The dimension ladder makes the trade legible. On S^1 the tick is one integer. On a genus-2 surface the same object is a winding *vector* in $\mathbb{Z}^4$. On the integer forest it is the winding of the climb. Same tick, read in different dimensions, always exactly computable, always Rule-9 free [exact re-encoding].

What the discount costs — and this is the chapter’s spine

There is no free invariant. Losing the dimension buys exactness, no analysis, no gauge, and a winding-number reading of the grading. It costs you **everything abelianization forgets**. Flat $U(1)$ connections on a surface are its Jacobian torus, which sees only H_1 — ordinary first homology, nothing a first course does not already see. The deep merchandise — Casson, Donaldson, exotic smooth structures — lives entirely in the *non-abelian* moduli, where the defining relation becomes a real constraint, the moduli space turns nontrivial, spectral flow becomes a genuine computation of a non-abelian Hessian, and the grading can collapse to $\mathbb{Z}/N$. That aisle is not stocked at abelian prices, and pretending otherwise is the one thing this chapter will not do.

And here is the kinship worth stating, because it is exact in kind: **this trade is this program’s own conservation of difficulty, one field up** [kinship of kind; Field Guide §8]. Every geometry forces one invariant and pays with exactly one blind spot; the blind spot is the receipt for the invariant. Novikov versus Floer is that law at the level of whole theories. The abelian corner forces an exact, computable grading and pays with blindness to everything non-abelian. Its ceiling and its exactness are the same fact — which is precisely what made the integer forest a *permanent* Flatland worth trusting: its group is abelian by nature, so it can never wander into the expensive aisle, and never has to.

Verdict

- Spectral flow = winding number on the abelian $U(1)$ ring — **[exact / theorem-backed]**, receipt `spectral_flow.py`.
- Circle-valued Morse = Novikov = Floer's finite-dim prototype and coefficient ring — **[settled]**.
- The forest / surface / ring are one torus-of-characters object seen in three dimensions — **[exact re-encoding]**, receipt `flatland_walk.py`.
- Novikov-vs-Floer is the conservation-of-difficulty law at the level of theories — **[kinship of kind]**; asserted as kind, not as a numerical bridge.

What losing a dimension buys you is an exact grading you can count on your fingers. What it costs you is the right to claim you have seen anything a homology group could not. Both halves are the chapter; a translation that reports only the first is the speculation this program fences out.

The Comma That Is a Row of Your Own Table

Where a bridge you reported negative turns out exact one aisle over

Crossover Atlas, Chapter 2. Matthew "Bear" Schulz · CC BY 4.0 · 2026-08-09. Receipt: 02_receipts/tower_comma.py.

The one-line finding, up front: the Pythagorean comma — the 2,000-year-old fact that twelve perfect fifths overshoot seven octaves — is *numerically identical* to row (19,12) of the Baker floor this program already measured. Two sections of the Tower paper describe the same object. This chapter draws the line between them.

The wall

Stack twelve perfect fifths, frequency ratio 3:2 each, and you expect to land seven octaves up. You don't. You overshoot by a hair that has haunted tuning since Pythagoras: **the comma**. Measured (`tower_comma.py`): $(3/2)^{12} \div 2^7 = 531441/524288 = 23.46$ cents. The loop of twelve fifths does not close. The gap is the comma.

That sentence should feel familiar, because it is this program's founding move verbatim: **a loop that fails to return, and the failure is the signal**. The comma is a closure deficit in the plainest possible sense — kinship of kind with holonomy, asserted at the level of *type* [kinship of kind].

The bridge the Tower paper tested — and correctly reported negative

The paper did the honest thing: it asked whether the comma's *value* bridges to any measured substrate constant — the 35.81° protein defect floor, the 0.1886 writhe, the DNA twist. It reported the bridge **negative**, and kept the story at kinship. That was correct. 23.46 cents is not 35.81 degrees, and forcing a number across would have been exactly the inheritance trap this atlas exists to catch.

But that was the wrong bridge to stop at. The comma does not bridge to the substrate constants. It bridges to the paper's own *Baker floor*.

The upgrade: it is row (19,12), exactly

Write the comma as a linear form in logarithms — which is what a ratio of prime powers is: $\text{comma} = 3^{12} / 2^{19}$, so its size is $|12 \cdot \ln 3 - 19 \cdot \ln 2| = \mathbf{1.355 \times 10^{-2}}$ (MEASURED). Now open the Baker-floor table from the Tower paper and read the third row: (a,b) = (19,12), value **1.355×10^{-2}** . The same number. The comma is that row [exact / theorem-backed].

And (19,12) is a convergent of $\log_2 3 = 1.58496\dots$ — a best rational approximation. That is why twelve is the famous near-miss: twelve fifths nearly close because 19/12 nearly equals $\log_2 3$. The *reason* twelve is the sweet spot, and the next real improvement waits until 53-tone equal temperament (the (84,53) row, 6.5× tighter), is the continued fraction of $\log_2 3$ — the same ladder of convergents the Baker floor is built from. To be precise about what is Baker's and what is trivial: the comma is *nonzero* for a trivial reason ($3^{12} \neq 2^{19}$, unique factorization). What is deep — how small the near-misses can get and how fast the ladder tightens — is the linear-forms-in-logarithms bound. Music has been standing on the Baker floor since Pythagoras and calling it the comma.

The atlas lesson this chapter teaches

A verdict can **upgrade**. "Bridge reported negative" against one target does not close a story; it closes *that* bridge. The discipline is not just to test the bridge and report it honestly — it is to notice you may have tested the wrong target. The comma sat at kinship-of-kind against the substrate constants and at *exact* against the Baker floor, and both readings are in the same paper. Chapter 1 (Novikov) was exact from the first test. This one had to find the right table first. Together they teach that the verdict is a property of the *pair* — the object and the target you bridge it to — not of the object alone.

Prior art — standing beside

The bridge in this chapter is not ours, and saying so is the point. That twelve fifths overshoot seven octaves, and that the good equal temperaments (12, 41, 53...) are the continued-fraction convergents of $\log_2 3$, is classical — Barbour, “Music and Ternary Continued Fractions,” *American Mathematical Monthly* 55(9), 545–555, and observed as far back as King Fang (~40 BC) for the 53-cycle. The comma’s standard name in modern tuning theory is the monzo **[–19 12]**: our “(19,12)” is the field’s own notation, not a coincidence we spotted. The quantitative floor — Baker’s theorem on linear forms in logarithms (Baker 1966; effective constants Matveev 2000) — is already cited in the Tower paper. What is ours is smaller and internal: the Tower paper carried the comma in one section and the Baker floor in another and never noticed they were the same row. This chapter draws that one line. Everything else belongs to the people above.

Verdict

- Comma = a closure deficit, a non-closing loop — **[kinship of kind]** with holonomy.
- Comma = $|12 \cdot \ln 3 - 19 \cdot \ln 2|$ = Baker-floor row (19,12) = the monzo **[–19 12]**; the good temperaments are convergents of $\log_2 3$ — **[settled / classical]**, Barbour 1948; receipt `tower_comma.py`.
- The internal observation that our own Tower paper’s comma section and Baker-floor section are one object — **[ours, internal]**, the only claim in this chapter.
- Comma vs substrate constants — **[bridge tested, reported negative]**, as the Tower paper already had it.

The Particle You Only Ever See By Its Absence

The neutrino, read by the deficit — the founding move, made literal

Crossover Atlas, Chapter 3. Matthew “Bear” Schulz · CC BY 4.0 · 2026-08-09. Receipt: 02_receipts/muon_neutrino_deficit.py.

Up front: the neutrino is the one particle present at every stage of the muon’s life, and it is never seen directly — only inferred from the books that fail to balance. That is this program’s founding move, made literal in particle physics: the deficit is the measurement. This chapter is kept, deliberately, at kinship of kind — the same *method*, not the same number.

The muon’s whole life is a neutrino story

Trace the particle we chased across these walls this month:

- **Birth.** A pion decays: $\pi^+ \rightarrow \mu^+ + \nu_\mu$. The muon is born alongside a muon neutrino, and that birth is where its spin polarization comes from.
- **Death, either way.** Decay: $\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$, two neutrinos. Capture: $\mu^- + p \rightarrow n + \nu_\mu$, one neutrino.

The muon was the loud particle. The neutrino was in the room the whole time — at the birth and at both deaths — and we never once counted it.

You read it by the deficit

You cannot catch the neutrino as it leaves. What you can do is add up the energy and momentum of everything you *did* see and notice the books don’t balance. The missing four-momentum is the neutrino’s only signature.

The receipt makes the deficit measurable (`muon_neutrino_deficit.py`). If the muon decayed to two bodies — an electron and one invisible partner — the electron would be **monoenergetic**, a single spike at 52.83 MeV. It is not. The decay electron comes out with a **continuous** spectrum from zero to 52.83 MeV, because *two* neutrinos share the leftover energy however they like. The shape of what you can see is the fingerprint of what you cannot. The continuity is the deficit.

And this is not a metaphor imported after the fact. Pauli postulated the neutrino in 1930 for exactly this reason: the continuous beta-decay spectrum implied energy was going missing, and rather than abandon conservation he read a real, undetected particle out of the gap. **The deficit was the measurement, and the particle was real** [settled]. Twenty-six years later it was caught directly. The gap came first.

The kinship, stated as kind and not as number

Neutrino-by-missing-momentum and holonomy-by-closure-deficit are the same *kind* of reading: infer the invariant from what fails to close, never from a direct magnitude. I tested for a numerical bridge and there is none to claim — a missing four-momentum is not a holonomy angle, and asserting an equation between them would be the inheritance trap. What transfers is the **method**, and it transfers at full strength: the neutrino is physics’s canonical proof that a null, read carefully, is not the absence of a measurement but the measurement itself.

That is the strongest possible endorsement of this program’s founding claim — that the deficit is the signal — and the honest way to keep it is exactly here, at kinship of kind. The comma upgraded to exact because a number matched. This one does not, and should not. A story kept at kinship on purpose is as much a part of the discipline as a bridge that turns out exact.

Verdict

- Neutrino present at the muon’s birth and both deaths — **[settled]**.
- The continuous decay spectrum is the fingerprint of the unseen neutrinos (vs a 2-body spike) — **[settled]**, receipt `muon_neutrino_deficit.py`.
- Pauli read a real particle out of an energy deficit — **[settled]**, the historical proof that a null is a measurement.
- Neutrino-by-deficit $\leftrightarrow$ holonomy-by-closure-deficit — **[kinship of kind]**, the method not the number; no numerical bridge claimed, and none exists.

The Rotors That Might Be the Zeros

A resemblance measured, a mechanism unproven — the conjectural chapter

Crossover Atlas, Chapter 4. Matthew “Bear” Schulz · CC BY 4.0 · 2026-08-09. Receipt: 02_receipts/cue_zeta.py, figure 04_figures/cue_vs_zeta.png.

Up front, and this is the whole point of the chapter: this is a connection the author has circled for months and wants to be true. It is kept at **[kinship of kind / conjectural]** anyway, because that is what is earned. The resemblance is measured. The mechanism that would make it more than a resemblance is unproven. The discipline is hardest exactly where you care most, and this chapter exists to prove the atlas holds it anyway.

The wall

The non-trivial zeros of the Riemann zeta function lie (conjecturally — this is the Riemann Hypothesis, unproven) on a single vertical line. A recurring dream since Hilbert and Pólya is that they are not scattered numbers but a **spectrum**: the eigenvalues of some self-adjoint operator, which would be real automatically and would place the zeros on the line for a *reason*. No such operator is known.

The image, and it is the right one

Picture sized rotors spread around a circle, pushing each other apart so they never collide. That is not a metaphor reaching for the zeros — it is the **Circular Unitary Ensemble**: the eigenvalues of a random unitary matrix, which lie on the unit circle and *repel*. Their nearest-neighbour spacings follow a specific law with almost no tiny gaps.

The measured resemblance

Montgomery (1973) computed the pair correlation of the zeta zeros; Odlyzko (1987) checked it numerically at enormous height. The zeros’ spacings match the CUE/GUE law. The receipt reproduces the core of it (`cue_zeta.py`): across the first 200 zeros and a CUE ensemble, the fraction of spacings below 0.3 is 0.00 for the zeros and 0.03 for the CUE — versus about 0.26 for a random (Poisson) process. Both **repel**, hard, the same way. [MEASURED]

The honest wrinkle, stated not hidden: the variance of the zero spacings comes out low (0.13 versus the ensemble’s 0.18), because the first 200 zeros sit at the bottom of the line where the limiting behaviour has not set in. Odlyzko had to go out near the 10^{20} -th zero to get the textbook-perfect match. The *law* is clearly shared; the *sharpness* is a function of height, and reporting the finite-size deviation is the difference between an atlas entry and a cherry-picked plot.

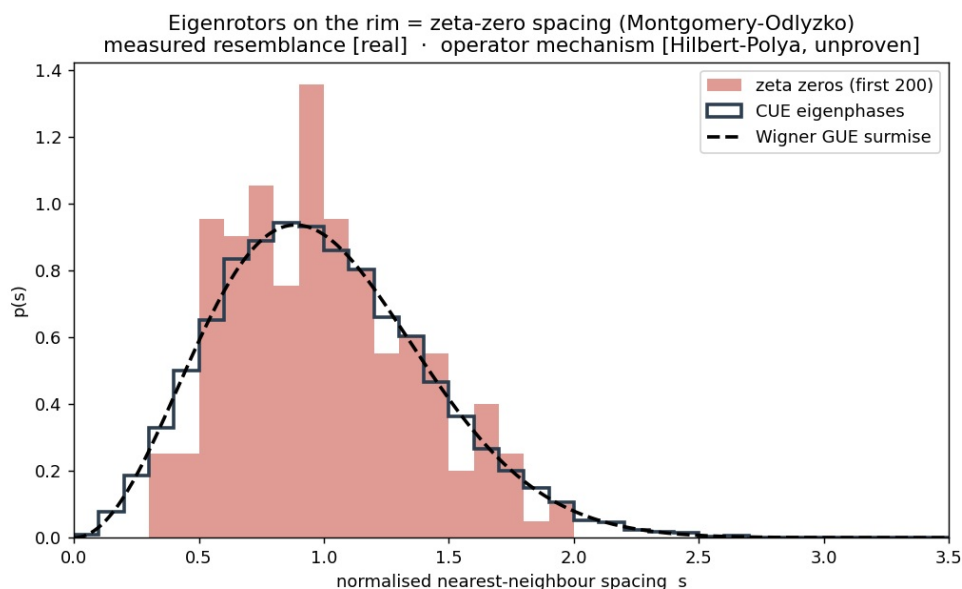


Figure 1. Nearest-neighbour spacings: the first 200 zeta zeros against the CUE law and a Poisson baseline — repulsion shared, sharpness a function of height. Generated by 02_receipts/cue_zeta.py.

The mechanism, and why it stays conjectural

That the spacings match is measured. That the zeros *are* eigenvalues of an operator — Hilbert-Pólya — is unproven. Berry and Keating proposed the operator might be a quantisation of the classical Hamiltonian xp , tying the zeros to a chaotic dynamical system; also conjectural. And RH itself is open. So the chain is: **resemblance, real; operator, hoped for; hypothesis, unproven**. The image is correct and it is not a proof of anything.

The verdict this chapter teaches

The other three chapters resolve: exact (Novikov), upgrades-to-exact (comma), held at kinship because no number should be forced (neutrino). This one carries the tag none of them do — **[conjectural]** — and it is the most important tag in the atlas, because it is the one that proves the collection is not just flattering connections. An atlas that only ever reported “exact” or “beautiful kinship” would be a sales brochure. The test of the method is whether it will look at the author’s own cherished idea and write *conjectural* on it in ink. It will.

Verdict

- Zeta-zero spacing = CUE/GUE spacing (repulsion) — **[settled / measured]**, Montgomery-Odlyzko, receipt `cue_zeta.py`.
- The zeros are the spectrum of a self-adjoint operator (Hilbert-Pólya) — **[conjectural]**, unproven.
- The x_p / Berry-Keating dynamical picture — **[conjectural]**, unproven.
- “Eigenrotors on the rim” = the CUE, the empirical face of the zeros — **[kinship of kind]**, the image is right; it proves nothing new, and RH remains open.

The Holonomy That Is Actually in the Sky

Where the instrument’s own object is measured across the whole universe — at the floor

Crossover Atlas, Chapter 5. Matthew “Bear” Schulz · CC BY 4.0 · 2026-08-09. Receipt: 05_spokes/T05_cosmic_birefringence_2026-08-09.md.

Up front: cosmic birefringence is not a metaphor for holonomy. It *is* one — the plane of the cosmic microwave background’s polarization, parallel-transported across 13.8 billion years and arriving rotated by an angle β . It is measured, it is real, and it sits right at the noise floor. Chapter 4 was a holonomy that *might* be there (the zeros). This is a holonomy that demonstrably *is*. That is the bracket.

The wall

As CMB photons travel from the last-scattering surface to us, a parity-violating interaction — an axion-like field coupled to electromagnetism through a **Chern-Simons** term — can rotate the plane of their linear polarization. The rotation angle is β . The question the whole field is asking is whether $\beta \neq 0$.

Why it is holonomy, exactly — and why it is a clean angle

β is the accumulated rotation of a vector transported along a path. That is a holonomy. And the Chern-Simons coupling makes it a **flat-connection** holonomy — which, by lesson 3, is exactly why β is a single clean number and not a shape-dependent mess: a flat connection’s holonomy depends only on the path’s topology, not its geometry. The sky is running your instrument’s continuous channel in the flat regime, at cosmological scale. [settled — it is holonomy by construction]

The measurement (current, cited)

- $\beta = 0.342^\circ (+0.094 / -0.091)$, 3.6σ , excludes $\beta=0$ at 99.987% CL — Eskilt & Komatsu 2022 (WMAP+Planck).
- $\beta \approx 0.30^\circ \pm 0.05^\circ$ — Planck legacy reanalysis.
- 3.3σ at $\beta = 0.33^\circ \pm 0.10^\circ$ from synchrotron-dominated data alone — Cosmoglobe DR1.
- Frequency-independent: consistent with an axion, disfavours Faraday rotation ($\beta \propto \nu^{-2}$).
- NOT yet a 5σ discovery. [measured]

The floor is a degeneracy, not noise — and that is the whole lesson

The dominant obstacle is not random noise. It is a **degeneracy**: the absolute miscalibration of each detector’s polarization angle adds directly onto β , and the naive estimator cannot tell them apart. Minami & Komatsu broke it by using the **Galactic foreground as a self-calibrator** — the foreground sits at redshift ~ 0 and sees no cosmological rotation, so it pins the instrument angle. The live systematic that remains is the intrinsic dust EB. So the competence-surface axis here is instrument-angle calibration $\div$ signal, with foreground EB as the confounder. A single-sigma quoted without the calibration model would be a fiction — which is this program’s own §11 lesson, appearing in someone else’s data.

The honest verdict, including what we do NOT add

- β is a Chern-Simons Wilson-loop rotation — **native holonomy**. [settled]
- The signal is real but **marginal / floor-limited**, $\sim 3.3\text{--}3.6\sigma$, not discovery. [measured]
- The floor is a **calibration degeneracy**, broken by foreground self-calibration; dust EB the live worry. [measured]
- **We add framing, not analysis**. The cosmology pipelines out-analyse any loop-walk we would bring. What the instrument's language contributes is the correct picture: β is a holonomy at the floor, and the floor is a degeneracy, not noise. This is a NATIVE target — a place the framework is *right*, not a place it *wins*. Saying so is the difference between an atlas and a claim.

Verdict

- Cosmic birefringence β = a flat-connection (Chern-Simons) holonomy in the sky — **[settled / native]**.
- $\beta \approx 0.3^\circ$, $\sim 3.3\text{--}3.6\sigma$, frequency-independent — **[measured]**, not yet discovery.
- The floor is a polarization-angle calibration degeneracy, broken by foreground self-cal — **[measured]**.
- Our contribution is competence-surface framing, not a better pipeline — **[honest scope]**.

The Integer That Guards Its Own Edge

The bulk-boundary correspondence — and the bigger chest found by not beelining

Crossover Atlas, Chapter 6. Matthew “Bear” Schulz · CC BY 4.0 · written 2026-08-22, completing the 2026-08-09 atlas. Receipt: 02_receipts/ssh_edge.py; spoke: 05_spokes/T11_topological_edge_states_2026-08-09.md.

Up front: Chapter 5 found the instrument's *continuous* channel running in the sky, real and floor-limited. This chapter finds the *integer* channel running in matter — exact, theorem-backed, and reproducible on a desk. Between them the instrument's two channels are each located in nature, not in analogy. And then, because the method's own rules say survey the domain before declaring victory, the chapter reports the larger find those rules turned up: the same integer, made of ocean and sky.

The wall

A topological insulator is a material that conducts only along its boundary, and the count of those boundary channels is not an engineering accident — it is fixed by an integer that lives in the material's *interior*. The cleanest case is the Su-Schrieffer-Heeger chain (1979): atoms in pairs, with two alternating hopping strengths, v inside each pair and w between pairs. The bulk of the chain carries a **winding number** — as the crystal momentum runs once around its circle, a two-component vector built from v and w either loops around the origin or it does not, and that is a 0-or-1 fact about the whole chain. The wall's claim, the **bulk-boundary correspondence**, is that this interior integer *equals* the number of protected zero-energy states pinned to the chain's ends.

The measurement: the gap is the abyss, and the integer flips only there

The receipt tunes the ratio w/v across the transition and reads three things at once — the winding, the energy gap of the bulk, and the count of zero-energy states on an open forty-cell chain (MEASURED, `ssh_edge.py`). Read the gap column first: at $w = 0.3$ the gap is 1.40; at 0.7 it is 0.60; at 0.95 it has thinned to 0.10; at $w = v$ it collapses to 0.0031, numerically the closed abyss; and on the far side it reopens, 0.10, then 1.00, then 4.00. The winding holds at 0 all the way to the abyss and is 1 immediately after it. That ordering is the theorem in one table: the integer *cannot* change while the gap is open, so the only place it flips is the one point where the gap closes. The gap resists, then breaks over, then the integer has flipped — a triac, made literal. And in the topological phase the open chain shows exactly **2** zero-energy edge states, one per end: the bulk integer, counted at the boundary.

The honest wrinkle, measured not hidden: just past the transition, at $w = 1.05$, the winding is already 1 but the forty-cell chain shows *zero* states within 10^{-6} of zero energy. Nothing is wrong with the theorem — the two edge states are there, but so close to the transition their tails stretch across the whole short chain, overlap, and split away from zero; by $w = 1.5$ they have locked to zero exactly. The correspondence is a statement about the long-chain limit, and a finite chain near the transition is the finite-size regime — the same kind of honesty Chapter 4 owed about the low-height variance of the zeta zeros. A table that omitted the 1.05 row would look cleaner and be a lie.

Why it is holonomy — the lexicon, so we know it is our object

The winding number of the bulk is a loop holonomy, under its condensed-matter name. It equals the **Zak phase** divided by π — and the Zak phase is nothing but the Berry phase of the Bloch state, parallel-transported once around the Brillouin zone’s circle. In words: the crystal’s momentum space is itself a loop, walking it accumulates a phase, and that phase read against π is the integer. One dimension up, the same object is the **Chern number** of TKNN (1982) — the integral of Berry curvature over the two-dimensional zone, the same monopole integer Chapter 8’s receipt counts on the sphere — and the correspondence (Hatsugai 1993) says that bulk integral equals the edge-mode count. Four names, one object: winding, Zak phase, Chern number, spectral flow. This is a mature field; every word of it belongs to the people in the References, and the atlas’s role is to point at the sameness, not to claim it.

The bigger chest: a Chern insulator made of ocean and sky

The method’s rules say: before celebrating the desk-sized hit, survey the domain list. Doing so turned up something larger sitting in this program’s own named territories. Delplace, Marston and Venaille (*Science*, 2017) showed that the shallow-water waves of a rotating planet carry a first **Chern number of 2** — Earth’s rotation breaks time-reversal symmetry, the Coriolis parameter changes sign at the equator, and the equator becomes a **domain wall** between two bulks whose integers differ. The bulk-boundary correspondence then *requires* protected edge modes riding that wall, and they are already famous under other names: the equatorial Kelvin and Yanai waves, which travel only eastward — unidirectional because they are topological, not because of any local forcing. The planet’s ocean and atmosphere are a Chern insulator, and the equator is its edge. The follow-ups extend the same reading to stratified fluids (Perrot, Delplace, Venaille 2019) and magnetized plasmas (Fu, Qin 2021). The beeline would have stopped at SSH and called it a day; the survey found the handle in the sand.

The discipline: “transition” is a false-friend word

Not every threshold is topology, and this chapter is where the tell gets stated: a *topological* transition has a protected integer that cannot change without a gap or order parameter closing; a mere threshold has no such integer. Melting is not topological. The Carnot cycle is a genuine *continuous* holonomy — the net work is the area the cycle encloses in state space, a loop integral in the plainest sense — but it is a magnitude, not an integer, and calling it more would be the inheritance trap. Vortex-unbinding transitions, by contrast, carry a real integer winding. The false friends are pinned here so a reader is told which is which.

Verdict

- SSH bulk winding = protected edge count, and the integer flips only where the gap closes (gap 1.40 $\rightarrow$ 0.0031 $\rightarrow$ 4.00 across the abyss; winding 0 $\rightarrow$ 1; 2 zero modes) — **[exact / theorem-backed]**, receipt 02_receipts/ssh_edge.py, finite-size wrinkle at $w = 1.05$ reported, not hidden.
- Winding = Zak phase / π = Berry phase around the Brillouin-zone loop; in 2D the Chern number (TKNN), with bulk integral = edge count (Hatsugai) — **[settled / classical]**.
- Equatorial Kelvin and Yanai waves are topologically protected edge modes of a planetary Chern number 2, with the equator as the domain wall — **[settled]**, Delplace-Marston-Venaille 2017; the larger find, in this program’s own listed domains.
- The “transition” tell — protected integer with a closing gap vs a mere threshold; Carnot = continuous holonomy (area = work), a magnitude, not an integer — **[discipline / false friends pinned]**.

The Restless Local and the Immovable Total

Gauss-Bonnet, Chern-Weil, and the theorem that fuses the program’s two halves

Crossover Atlas, Chapter 7. Matthew “Bear” Schulz · CC BY 4.0 · written 2026-08-22, completing the 2026-08-09 atlas. Receipts: 02_receipts/gauss_bonnet.py, 02_receipts/clustering_topology.py.

Up front: every chapter so far has read an invariant in some field’s own language. This chapter is about the theorem that explains why the reading works at all — why a quantity you compute from restless local choices can come out as an integer that no choice can move. It is also where this program found its first strong handhold: “read an invariant, not a magnitude” was learned here, on this theorem, before it was a motto.

The wall

Curvature is the most local thing a surface owns, and nobody can hold it still. Press a dent into a sphere and the curvature concentrates at the rim of the dent, thins across the flattened middle, and every number you wrote down before is wrong. It depends on the shape, on the metric, on every choice made in the description. It is the purest possible **magnitude** — and the founding wager of this whole program is that magnitudes are the wrong thing to read.

The translation: total the restless thing and it cannot move

The Gauss-Bonnet theorem says: integrate the curvature over a whole closed surface and the answer is 2π times an integer — the Euler characteristic χ — and that integer does not care about any of the choices. The receipt measures this in the bluntest way available (`gauss_bonnet.py`). Take a cube, where all the curvature sits at the eight corners as angle defect: the defects total 12.56637 radians, which is 4π exactly, so $\chi = 2$. Now deform the cube — stretch it, break its symmetry, move its curvature somewhere else entirely — and total again: still 12.56637, still 4π , χ still 2. The curvature moved; the integer did not. Then do the torus: total angle defect 0, so $\chi = 0$. One integer, and it tells the sphere from the donut while every local number is busy changing.

And here is the bridge, in the same receipt: count the torus and sphere the *other* way, by their holes. The alternating sum $b_0 - b_1 + b_2$ — components, minus independent loops, plus enclosed voids — gives $1 - 0 + 1 = 2$ for the sphere and $1 - 2 + 1 = 0$ for the torus. **The same integers.** A quantity born as an integral of curvature and a quantity born as a count of holes agree, number for number [exact, MEASURED]. That is not a coincidence to admire; it is the theorem doing exactly what it is for.

Chern-Weil, plainly — why this is the whole instrument

Stated in words, the Chern-Weil recipe is three steps. Pick any connection — any rule for parallel-transporting, any metric, any of the restless choices. Compute its curvature, which is the local holonomy density: what an infinitesimal loop reads. Then integrate the right expression built from that curvature over the closed manifold — and the result is a topological integer (Euler, Chern) that is *independent of the choice made in step one*. The curvature is connection-dependent; its integral is not. A magnitude goes in; an invariant comes out; the choices cancel in the total. “Read an invariant, not a magnitude” is not this program’s slogan pasted onto geometry — it *is* Chern-Weil, and the program learned the discipline from the theorem, not the other way around. Chapter 6’s Chern number and Chapter 8’s monopole integer are both this recipe running in their own fields.

And the bridge fuses the program’s two halves. One half of the instrument walks loops and reads holonomy; the other half counts — clusters, holes, persistent features in data. Gauss-Bonnet is the theorem that says these were never two instruments: the loop-reading integrates to χ , and χ is the hole-count $b_0 - b_1 + b_2$. The walk and the count are the same integer, arrived at from opposite ends.

The count channel, measured on data

The counting half deserves its own receipt, on distributions rather than surfaces (`clustering_topology.py`). Sample a noisy circle and grow a neighborhood radius r through it, tracking the Betti numbers as you go: the count $(b_0, b_1) = (1, 1)$ — one piece, one hole — holds across the whole range $r \in [0.24, 0.30]$. A random blob of the same size never grows a persistent hole at all; its short-lived features are born and die immediately. Three planted clusters hold $b_0 = 3$ across the wide range $r \in [0.31, 0.89]$ before merging. **Persistence is the integer’s signature:** a real topological feature survives across a range of scales, and noise dies young — which is the count-channel’s version of “the invariant doesn’t move when the choices do,” with the radius r playing the role of the choice. This is standard persistent homology (Edelsbrunner-Harer; Carlsson; Ghrist), used here exactly as its builders built it. The same receipt carries one small aside from this program’s own history: six symmetry-related crystal axes collapse their fifteen possible pairwise distances onto just three values — a count-collapse that flags hidden symmetry, the counting channel reading a crystal the way the walk reads a loop.

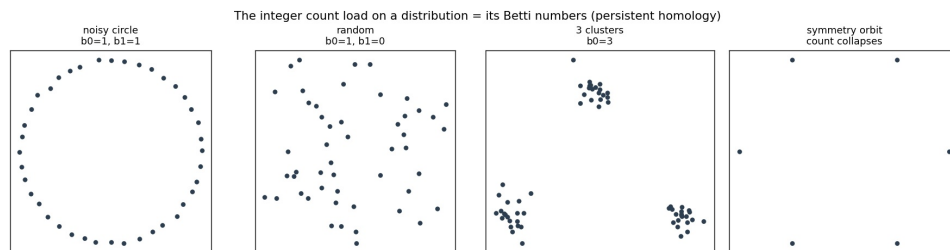


Figure 2. The count channel: the noisy circle holds $(b_0, b_1) = (1, 1)$ across a range of scales and the three clusters hold $b_0 = 3$, while the blob’s features die young. Generated by `02_receipts/clustering_topology.py`.

Standing beside

Gauss-Bonnet is classical differential geometry; its lattice form — curvature as angle defect at the corners, exactly what the cube receipt totals — is Descartes' theorem, older than the smooth version. Chern-Weil theory is the standard machinery of characteristic classes. Persistent homology belongs to Edelsbrunner and Harer, Carlsson, and Ghrist, with the statistics of random complexes to Kahle and Bobrowski-Kahle. Nothing in this chapter is ours except the assembly — and the debt, which this chapter exists to record: the program's founding discipline was learned from these theorems.

Verdict

- Total curvature = $2\pi\chi$, invariant under deformation: cube and deformed cube both 4π ($\chi = 2$), torus 0 ($\chi = 0$), while the local curvature moves freely — **[exact]**, receipt `02_receipts/gauss_bonnet.py`.
- χ from curvature = χ from counting holes: $2 = 1 - 0 + 1$ and $0 = 1 - 2 + 1$, measured both ways — **[exact / the bridge]**; this is the theorem that fuses the loop-walk and the count into one instrument.
- Chern-Weil: connection-dependent curvature integrates to connection-independent integers; “read an invariant, not a magnitude” is this theorem — **[settled / classical]**.
- Persistent Betti numbers are the count channel's invariant, and persistence across scale is the test that separates a real feature from noise (circle (1,1) held; blob's hole never lived; $b_0 = 3$ held wide) — **[settled / TDA]**, receipt `02_receipts/clustering_topology.py`.

The One Number Four Fields Already Share

Girard's excess = area = solid angle = holonomy = Berry phase — the textbook chapter, and it claims nothing

Crossover Atlas, Chapter 8. Matthew “Bear” Schulz · CC BY 4.0 · written 2026-08-22, completing the 2026-08-09 atlas. Receipts: `02_receipts/spherical_excess.py`, `02_receipts/berry_monopole.py`.

Up front: this is the atlas's whole thesis stated once and settled by a single number. Draw a triangle on a sphere. Four fields hold five names for one quantity attached to it — the *excess* of its angles, the *area* it encloses, the *solid angle* it subtends, the *rotation* a vector suffers carried around it, and the *Berry phase* a quantum spin acquires steered around it — and all five are the same number. Not five facts that rhyme. One fact, five names, four alphabets. And every piece is so classical that claiming any of it would be absurd, which is exactly why this is the right chapter to close on.

The wall

A triangle on a flat table has angles that sum to exactly half a turn — to π . A triangle drawn on a sphere, with sides that are arcs of great circles, does not: its angles sum to *more* than π , and the amount of the overshoot grows with how big the triangle is. That overshoot is called the spherical **excess**, and it is the wall this chapter reads. The claim is that the excess is not a curiosity of the drawing — it is the same quantity that the loop-walk and the quantum spin have been measuring all along.

Girard: the overshoot is the area

Girard's theorem, from the early 1600s, says the excess *equals the area* the triangle covers on the unit sphere. In words: the amount by which the three angles overrun a flat triangle's π is exactly the patch of sphere the triangle fences in. A tiny triangle barely overshoots and encloses almost nothing; a triangle covering an eighth of the sphere overshoots by a right angle. On a sphere, angle and area are the same book kept two ways.

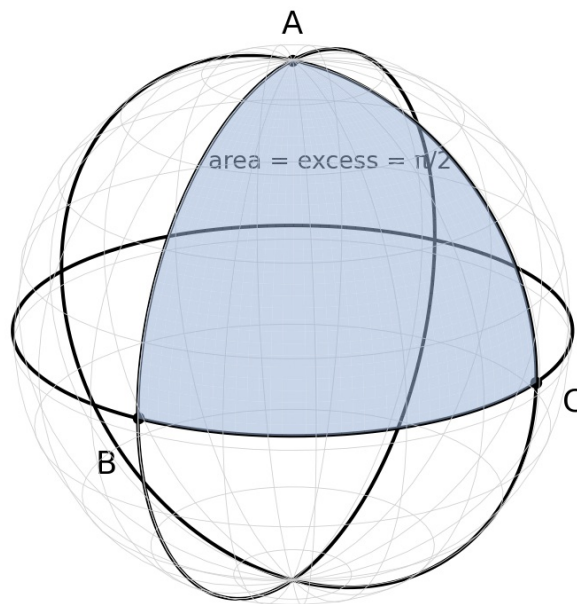


Figure 3. Three great circles cut the unit sphere into eight spherical triangles; the shaded octant has three right angles, so its excess — and therefore its area — is $\pi/2$. Deform the circles and the eight areas trade while their total holds at 4π . Generated by 02_receipts/fig_girard_octant.py.

The receipt (`spherical_excess.py`) computes that area three different ways and watches them agree. The first way is Girard’s — add the three corner angles and subtract π . The second is **L’Huillier’s theorem**, the spherical version of Heron’s formula: it recovers the same area from the three *side-lengths* alone, folded through a tangent, so you never measure an angle at all. The third is the **Van Oosterom-Strackee** formula, which recovers it from the three *corner directions* as vectors, also through a tangent — the form used in electrocardiography to read the solid angle a heart’s dipole subtends at an electrode. Three roads — from the angles, from the sides, from the corners — and for the right-angled “octant” triangle all three return the identical number, 1.57080 radians, which is $\pi/2$. For a second, lopsided triangle they again agree, at 0.62044. Same area, measured from whatever you happen to hold.

The loop-walk: that area is the holonomy

Now carry a vector around that same triangle, keeping it as parallel to itself as the curved surface allows — the parallel transport the loop-walk runs on. It comes home **rotated**, and nothing turned it but the shape of the path. The angle it returns rotated by is, exactly, the area enclosed. So the geometer’s “excess” and the instrument’s “holonomy” are not analogues; they are one number. The receipt confirms it: the vector transported around the octant comes home turned by 1.57080 radians — the same $\pi/2$ the three area formulas produced. What the loop-walk reads as a failure to close is what Girard reads as an area.

The quantum spin: that holonomy is the Berry phase

Steer a quantum spin — or any two-level system — slowly around the same loop of directions, and when it returns its state carries a leftover **phase**, remembered from the trip. That phase is the same solid angle, taken at *half* strength. In words: the quantum state records the loop it was carried around, and the record is the enclosed area, halved. The receipt gives the octant’s Berry phase as 0.78540 radians, which is $\pi/4$ — exactly half of the $\pi/2$ the geometry produced. The halving is not a fudge; it is the signature of a spinor, and the next paragraph is where it comes from.

The crossover point, and the 720° turn

Two things sharpen the quantum reading, and both are in `berry_monopole.py`. First, the phase we have been calling geometric becomes *topological* the moment the loop encircles a **degeneracy** — a point where the two levels cross, a Berry monopole. A single small loop still reads any smooth value, but sum the phases over a *closed* sphere that surrounds the crossing and the total is forced to a whole number: the Chern number, which the receipt returns as 1, the charge of the crossing inside. The local phase can be anything; the global sum counts an integer. That is the same integer Chapter 6 measured at the chain’s ends and the same “local density integrates to a whole number” that Chapter 7 runs on — arrived at here from a spinning compass instead of a lattice.

Second, the halving. A loop that fences off 2π of solid angle — a great circle, a full 360° turn of the guiding field — brings the spinor home multiplied by -1 , not $+1$. To return it honestly to $+1$ you must go around *twice*: 4π of solid angle, a **720°** turn. The receipt shows exactly this — 2π of solid angle gives -1 , 4π gives $+1$. That doubled turn is the reason the Berry phase carried half the area rather than all of it: a spin lives on the double cover of the sphere of directions, so it counts turns in halves.

Standing beside

This chapter is the purest case of the atlas’s whole posture, because it invents nothing whatsoever. Girard’s theorem is four centuries old. L’Huillier’s tangent form is eighteenth-century. Van Oosterom and Strackee published their vertex formula in 1983, and it was old news in geometry long before that. The parallel-transport reading is standard differential geometry. Berry named the geometric phase in 1984, and Pancharatnam had the optical version in 1956, decades earlier. Every wall here was built by someone named in the References, and the 720° spinor was understood long before any of them. The only thing this chapter does is stand the five readings next to each other and point at the single number underneath. That is the contribution, and it is deliberately nothing but the map.

The lesson this chapter teaches, and why it closes the atlas

Every earlier chapter argued “one object, two languages” for a wall where the sameness took real work to see — where a bridge had to be tested, or held at kinship, or marked conjectural. Here the sameness is a theorem so settled that no test is needed and no claim is possible. That is the point of ending on it. If the whole atlas reduces to one sentence, it is this: a quantity read in geometry as an *area*, in spherical trigonometry as an *excess*, in the loop-walk as a *holonomy*, and in quantum mechanics as a *Berry phase* is not four coincidences — it is one fact written in four alphabets, and the index theorems — Gauss-Bonnet the first of them, the Atiyah-Singer theorem their summit — are the dictionary that proves the translations exact. Every chapter of this atlas paired an integral along a walk with a count of crossings and found them equal; Atiyah-Singer is the general theorem with exactly that shape, and walking toward it is where this map continues. The atlas spent seven chapters earning the right to say that plainly. This chapter is where it gets to.

Verdict

- Girard’s excess = area = solid angle = parallel-transport holonomy = Berry phase (octant: all $1.57080 \text{ rad} = \pi/2$; spin- $\frac{1}{2}$ phase = half that, $\pi/4$) — **[settled / textbook]**, receipt `spherical_excess.py` + `berry_monopole.py`.
 - Three independent formulas — from the angles (Girard), the sides (L’Huillier), the vertices (Van Oosterom-Strackee) — return the identical number — **[exact]**, receipt `spherical_excess.py`.
 - A loop enclosing a degeneracy $\rightarrow$ the closed-sphere total is the Chern number 1 (monopole charge); a spinor needs 720° to return to $+1$ — **[settled]**, receipt `berry_monopole.py`.
 - The chapter’s own contribution is the exposition alone; every result is classical and cited — **[ours: nothing but the map]**.
-

References

Canonical prior art, by chapter. Author-year and venue are given where certain; volume/page details, where stated, were verified against publisher records on 2026-08-22; a credentialed zbMATH / MathSciNet pass remains pending. Any located hit not below will be added in the open with attribution.

Chapter 1 — Floer / Novikov. Novikov, S. P. (1981), multivalued functions and functionals (Novikov homology). Floer, A. (1988), Morse theory for Lagrangian intersections / instanton homology. Farber, M. (2004), *Topology of Closed One-Forms*. Atiyah, M., Patodi, V., Singer, I. (1975-76); Robbin, J., Salamon, D. (1993) — spectral flow and the Maslov/Floer index.

Chapter 2 — the comma. Barbour, J. M. (1948), “Music and Ternary Continued Fractions,” *American Mathematical Monthly* 55(9), 545–555. Baker, A. (1966), linear forms in logarithms; Matveev, E. M. (2000), effective lower bounds. Comma as the monzo [−19 12] (standard tuning-theory notation); convergents of $\log_2 3$, OEIS A028507; the 53-cycle noted by King Fang (~40 BC) and Mercator.

Chapter 3 — the neutrino. Pauli, W. (1930), the “radioactive ladies and gentlemen” letter proposing the neutrino to conserve energy-momentum in β -decay.

Chapter 4 — CUE / zeta. Montgomery, H. L. (1973), pair correlation of zeta zeros. Odlyzko, A. M. (1987), numerical statistics of zeta zeros. Berry, M. V., Keating, J. P. (1999), the $H = xp$ picture. Katz, N., Sarnak, P. (1999); Keating, J. P., Snaith, N. C. (2000), random-matrix statistics of L-functions.

Chapter 5 — cosmic birefringence. Carroll, S., Field, G., Jackiw, R. (1990); Harari, D., Sikivie, P. (1992); Carroll, S. (1998) — Chern-Simons / axion birefringence. Minami, Y., Komatsu, E. (2020); Eskilt, J. R., Komatsu, E. (2022) — measurements ($\beta \approx 0.3^\circ$).

Chapter 6 — edge states (from target-bank T11 / T12). Su, W. P., Schrieffer, J. R., Heeger, A. J. (1979) — the SSH model. Thouless, D., Kohmoto, M., Nightingale, M., den Nijs, M. (1982) — the TKNN / Chern invariant. Hatsugai, Y. (1993) — bulk-boundary correspondence. Delplace, P., Marston, J. B., Venaille, A. (2017), “Topological origin of equatorial waves,” *Science* 358(6366), 1075–1077. Zak (1989), Berry’s phase for energy bands in solids. Perrot, M., Delplace, P., Venaille, A. (2019), “Topological transition in stratified fluids,” *Nat. Phys.* 15, 781–784. Fu, Y., Qin, H. (2021), “Topological phases and bulk-edge correspondence of magnetized cold plasmas,” *Nat. Commun.* 12, 3924.

Chapter 7 — Clustering / TDA and the Gauss-Bonnet / Chern-Weil bridge. Edelsbrunner, H., Harer, J. (2010), *Computational Topology*. Carlsson, G. (2009), topology and data; Ghrist, R. (2008), barcodes. Kahle, M. (2011); Bobrowski, O., Kahle, M. (2018) — random / stochastic topology. Chern-Weil theory and the Gauss-Bonnet theorem (standard differential geometry).

Chapter 8 — Girard / Berry. Berry, M. V. (1984), “Quantal phase factors accompanying adiabatic changes,” *Proc. Roy. Soc. A* 392(1802), 45–57. Girard’s theorem (spherical excess = area = solid angle); Pancharatnam, S. (1956); L’Huillier’s theorem; Van Oosterom, A., Strackee, J. (1983), “The Solid Angle of a Plane Triangle,” *IEEE Trans. Biomed. Eng.* BME-30(2), 125–126. Vanderbilt, D. (2018), *Berry Phases in Electronic Structure Theory*.

Standing Beside note: our only genuine contribution across these chapters is the assembly and the verdicts — walking the loop that connects landmarks others planted, and reporting, at each one, exactly what kind of true it is.